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A REFINED TWO-DIMENSIONAL THEORY FOR THICK CYLINDRICAL SHELLS

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Abstract—A higher order two-dimensional theory for thick cylindrical shells is presented in this paper. The shell equations given here do not only incorporate the effect of transverse shear deformations but also account for the initial curvature as well as the radial stress. The proposed theory presents a very good approximation for the shell constitutive equations and the nonlinear distributions of the in-plane stresses across the thickness of the shell. The latter is very important in the thick shells analysis. The formulation is based on: (1) assumed out-of-plane stress components which satisfy the given traction boundary conditions; (2) three-dimensional elasticity equations with an integral form of the equilibrium equations; and (3) stress resultants and stress couples acting on the middle surface of the shell, average displacements along the normal at a point on the middle surface, and average rotations of the normal.

The proposed shell equations can be conveniently used in the finite element analysis. An application of this theory to the finite element analysis of circular arches is given in this paper. A more convenient form of the proposed shell equations for finite element analysis and its application to cylindrical shells will be presented in a follow-up paper.

INTRODUCTION

Although the complete two-dimensional linear theory of thin shells was developed by Love 100 years ago, numerous contributions since then have been made to this subject. This is primarily because any two-dimensional theory of shells is an approximation to the real problem. Researchers have been seeking better approximations for the exact three-dimensional elasticity solutions of shells. In the last three decades, the developed refined two-dimensional linear theories of thin shells include the important contributions of Sanders (1959), Koiter (1960), Flugge (1960) and Niordson (1978). In these refined theories, the initial curvature effect is taken into consideration in the formulation of the shell equations. Nevertheless, the deformation is based on the Love–Kirchhoff assumption, and the radial stress effect is neglected. These refined theories provide very good results for the analysis of thin shells. The theory of Sanders–Koiter has been widely used in the finite element analysis of shells (Ashwell and Gallagher, 1976).

However, it is shown (see Niordson, 1971) that Love's strain energy expression has inherent errors of relative order $[h/R+(h/L)^2]$ where h is the thickness of the shell, R is the magnitude of the smallest principal radius of curvature, and L is a characteristic wave length of the deformation pattern of the middle surface. Consequently, when the refined theories of thin shells are applied to thick shells, that is h/R is not small compared to unity, the error could be quite large as expected. Relative to the theory of thin shells, the theory of thick shells has received limited attention by the researchers up to now. With the increase of the utilization of thick shells to various engineering applications such as cooling towers, arch dams, pressure vessels, etc. it is imperative to develop a simple and accurate theory for thick shells.

Thick shells have a number of distinctly different features from thin shells. One of these features is that in thick shells the transverse shear deformation may no longer be neglected. In a number of particular cases of loadings the radial stress distribution of thick shells is very important and needs to be incorporated in the shell analysis. A third important distinction between thick and thin shell analyses is that in thick shell analysis the initial curvatures do not only contribute to the stress resultants and stress couples, but also result in nonlinear distributions of the in-plane stresses along the thickness of the shell.

It is not difficult to incorporate transverse shear deformations in shells. This can be accomplished following the work of Reissner (1945) for the plate theory. Nevertheless, it is not an easy task to incorporate radial stresses in thin shell theory and to obtain nonlinear stress distributions through the shell thickness in order to describe the behavior of thick shells. The attention in the previously developed shell theories was focused on the two-dimensional shell equations together with maintaining a linear stress distribution through the shell thickness (see Flugge, 1960; Niordson, 1985). It appears that refinement of the stress distribution in the thick shells has hardly been ignored. The theory of thin shells may provide a good estimate of the strain energy for some problems in thick shells. Nevertheless, it cannot provide an accurate distribution for the stresses through the thickness (Gupta and Khatua, 1978). This accuracy is imperative from an engineering point of view.

The formulation procedure for the proposed shell theory is based on the following:

(1) assumed out-of-plane stress components that satisfy the given traction boundary conditions;

(2) three-dimensional elasticity equations with an integral form of the equilibrium equations;

(3) stress resultants and stress couples acting on the middle surface of the shell together with average displacements along a normal of the middle surface of the shell and the average rotations of the normal (Voyiadjis and Baluch, 1981).

Although the proposed shell theory in this work is limited to thick cylindrical shells, the methodology can be extended to general shells by considering the general geometry of the shell.

It is well established that curved beams exhibit a nonlinear circumferential stress distribution through the thickness. In the proposed theory of shells, all the in-plane stresses exhibit a nonliner distribution through the thickness. This is primarily due to the incorporation of the initial curvature effect in the theoretical formulation of the proposed shell theory. The nonlinear stress expressions given here are compared for specific examples to those obtained through the three-dimensional theory of elasticity.

Due to the incorporation of the initial curvature of the shell in the proposed theory, we note that neither the stress resultant tensor nor the stress couple tensor is symmetric. The resulting constitutive equations of shells reduce to those given by Flugge (1960) when the shear deformation and radial effects are neglected. In this case the average displacements are replaced by the middle surface displacements. However, the resulting equations are slightly different from those given by Sanders (1959), Koiter (1960) and Niordson (1978). This is primarily because the so-called effective stress tensor and effective moment tensor are used in the derivation of the constitutive equations instead of the usual stress tensors (Niordson, 1971).

The proposed shell equations can be conveniently used in the finite element analysis. An application of this theory to the finite element analysis of circular arches is given in this paper. Numerical results are obtained and compared with the elasticity solution.

THEORETICAL FORMULATION OF THE REFINED THEORY OF THICK CYLINDRICAL SHELLS

Displacement field

The proposed theory for thick circular cylindrical shells incorporates the effects of the initial curvature and radial stress in addition to the effect of transverse shear deformations. The following kinematic field for u, v and w displacements along the x, ϕ and z direction is defined [see eqns (7)-(17)], respectively, for isotropic linear elastic materials:

$$u(x,\phi,z) = u_0(x,\phi) - z\frac{\partial w_0}{\partial x} + \frac{2v}{Eh^3}\frac{\partial M}{\partial x}z^3 + \frac{Q_x}{2Gh}z\left[\left(3-\frac{4z^2}{h^2}\right)-\frac{1}{R}\left(\frac{3z}{2}-\frac{3z^3}{h^2}\right)\right]$$
$$-\frac{1}{Ec_1}\frac{\partial p_i}{\partial x}\left[\frac{z^2}{2}-\frac{r_2^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right]-\frac{1}{Ec_2}\frac{\partial p_o}{\partial x}\left[\frac{z^2}{2}-\frac{r_1^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right]$$
$$+\frac{p_{xi}}{Gc_1}\left[z-\frac{r_2^2}{R^2}\left(z-\frac{z^2}{R}\right)\right]+\frac{p_{xo}}{Gc_2}\left[z-\frac{r_1^2}{R^2}\left(z-\frac{z^2}{R}\right)\right] \quad (1)$$

$$v(x,\phi,z) = \left(1+\frac{z}{R}\right)\left\{v_0(x,\phi)+\frac{Q_\phi}{2Gh}z\left[\left(3-\frac{4z^2}{h^2}\right)-\frac{3z}{R}\left(\frac{1}{2}-\frac{z^2}{h^2}\right)\right]\right.$$
$$-\frac{1}{R}\frac{\partial w_0}{\partial\phi}\left(z-\frac{z^2}{R}\right)+\frac{2\nu}{Eh^3}\frac{1}{R}\frac{\partial M}{\partial\phi}z^3\left(1-\frac{3z}{4R}\right)$$
$$-\frac{1}{Ec_1}\frac{1}{R}\frac{\partial p_i}{\partial\phi}\left\{\frac{z^2}{2}-\frac{r_2^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)-\frac{2}{R}\left[\frac{z^3}{3}-\frac{r_2^2}{R^2}\left(\frac{z^3}{3}-\frac{z^4}{4R}\right)\right]\right\}$$
$$-\frac{1}{Ec_2}\frac{1}{R}\frac{\partial p_o}{\partial\phi}\left\{\frac{z^2}{2}-\frac{r_1^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)-\frac{2}{R}\left[\frac{z^3}{3}-\frac{r_1^2}{R^2}\left(\frac{z^3}{3}-\frac{z^4}{4R}\right)\right]\right\}$$
$$+\frac{p_{\phi i}}{Gc_1}\left[z-\frac{r_2^2}{R^2}\left(z-\frac{z^2}{R}\right)-\frac{1}{2R}\left(z^2-\frac{r_2^2}{R^2}\left(z^2-\frac{4}{3}\frac{z^3}{R}\right)\right)\right]$$
$$\left.+\frac{p_{\phi o}}{Gc_2}\left[z-\frac{r_1^2}{R^2}\left(z-\frac{z^2}{R}\right)-\frac{1}{2R}\left(z^2-\frac{r_1^2}{R^2}\left(z^2-\frac{4}{3}\frac{z^3}{R}\right)\right)\right]\right\} \tag{2}$$

$$w(x,\phi,z) = w_0(x,\phi)+\frac{1}{E}\left\{\frac{p_i}{c_1}\left[z-\frac{r_2}{R^2}\left(z-\frac{z^2}{R}\right)\right]+\frac{p_o}{c_2}\left[z-\frac{r_1}{R^2}\left(z-\frac{z^2}{R}\right)\right]-\nu M\frac{6z^2}{h^2}\right\}, \tag{3}$$

where

$$M = M_x+M_\phi \tag{4}$$

and

$$c_1 = \left(\frac{r_2}{r_1}\right)^2-1 \tag{5}$$

$$c_2 = 1-\left(\frac{r_1}{r_2}\right)^2. \tag{6}$$

In eqns (1)–(3), $u_0(x,\phi)$, $v_0(x,\phi)$ and $w_0(x,\phi)$ are the displacements in the x, ϕ and z directions, respectively, of the mid-surface $z = 0$. The loads p_{xi} and p_{xo} are distributed along the x direction on the inner and outer surfaces, respectively, and similarly for the loads $p_{\phi i}$ and $p_{\phi o}$ along the ϕ direction. In the above expressions, E and ν are the Young's modulus and Poisson's ratio, respectively, h is the thickness of the shell, and $p_i(x,\phi)$ and $p_o(x,\phi)$ are the radial loads exerted on the shell surfaces $z = -h/2$ and $z = h/2$, respectively. The positive sign convention for these loads is shown in Fig. 1. Q_x and Q_ϕ are the transverse shears, and M_x and M_ϕ are the moment stress resultants on the planes of x = constant and ϕ = constant, respectively.

The addition of the transverse normal strain effect has resulted in a transverse displacement w whose distribution through the thickness is explicitly obtained on physical grounds as a nonlinear function in z. The detailed derivation of the displacement field is outlined below.

The following out-of-plane stress components are assumed:

$$\sigma_z = \frac{1-(r_2/r)^2}{c_1}p_i+\frac{1-(r_1/r)^2}{c_2}p_o, \tag{7}$$

$$\tau_{xz} = \left(1-\frac{z}{R}\right)\left(1-\frac{4z^2}{h^2}\right)\frac{3Q_x}{2h}+\frac{1-(r_2^2/r^2)}{c_1}p_{xi}+\frac{1-(r_1^2/r^2)}{c_2}p_{xo}, \tag{8}$$

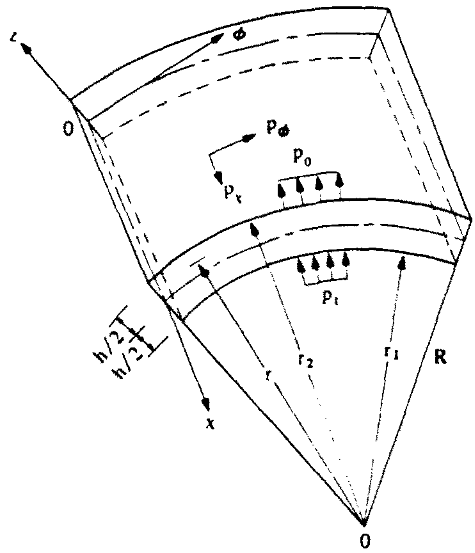

Fig. 1. Cylindrical shell element.

$$\tau_{\phi z} = \left(1 - \frac{4z^2}{h^2}\right)\frac{3Q_\phi}{2h} + \frac{1-(r_2^2/r^2)}{c_2}p_{\phi i} + \frac{1-(r_1^2/r^2)}{c_2}p_{\phi o}. \tag{9}$$

Expression (7) depicts the radial stress distribution as obtained from the elasticity solution for thick cylinders subjected to constant radial loads at both surfaces $z = h/2$ and $z = -h/2$. The normal stress σ_z is ignored in the case of analysis of thin shells. Equation (9) expresses the transverse shear stress as obtained from rectangular cross-sections. In the case of eqn (8), the transverse shear on the surface x = constant is modified through the term $(1-z/R)$ due to the fact that the cross-section is not rectangular in this case but exhibits a curvature. Equations (7)–(9) satisfy the following boundary conditions

$$\sigma_z = p_o \quad \text{at } z = h/2 \tag{10a}$$

$$\sigma_z = -p_i \quad \text{at } z = -h/2 \tag{10b}$$

$$\tau_{z\phi} = p_{\phi o} \quad \text{at } z = h/2 \tag{10c}$$

$$\tau_{z\phi} = -p_{\phi i} \quad \text{at } z = -h/2 \tag{10d}$$

$$\tau_{xz} = p_{xo} \quad \text{at } z = h/2 \tag{10e}$$

$$\tau_{xz} = -p_{xi} \quad \text{at } z = -h/2. \tag{10f}$$

The assumed stress field satisfies the weak form of the equilibrium equation given by the following integral expression

$$\int_{-h/2}^{h/2}\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{\phi z}}{r\,\partial\phi} + \frac{\partial\sigma_z}{\partial z} + \frac{\sigma_z - \sigma_\phi}{r}\right)\mathrm{d}z = 0. \tag{11}$$

Using Hooke's law for a linear elastic material, we obtain the transverse normal strain ε_z in terms of the stresses:

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_\phi)]. \tag{12}$$

The sum $(\sigma_x + \sigma_\phi)$ is assumed to be given by the following expression:

$$\sigma_x + \sigma_\phi = \frac{12(M_x + M_\phi)z}{h^3}. \tag{13}$$

Reissner (1975) considered a plate with no load and made use of expression (13) to modify the expression for the transverse displacement w. Substituting expressions (7) and (13) into eqn (12), we obtain

$$\frac{\partial w}{\partial z}=\frac{1}{E}\left[\frac{1-(r_2/r)^2}{c_1}p_i+\frac{1-(r_1/r)^2}{c_2}p_o-\frac{12v}{h^3}(M_x+M_\phi)z\right]. \tag{14}$$

Integrating eqn (14) with respect to z yields the following expression for the displacement w:

$$w(x,\phi,z)=w_0(x,\phi)+\frac{1}{E}\left\{\int\left(\frac{1-(r_2^2/r^2)}{c_1}p_i+\frac{1-(r_1^2/r^2)}{c_2}p_o\right)\mathrm{d}z-v\frac{6z^2}{h^3}M\right\}$$

or

$$w(x,\phi,z)=w_0(x,\phi)+\frac{1}{E}\left\{\frac{p_i}{c_1}\left[z-\frac{r_2^2}{R^2}\left(z-\frac{z}{R}+\frac{z^2}{R^2}-\frac{z^3}{R^3}+\ldots\right)\right]\right.$$

$$\left.+\frac{p_o}{c_2}\left[z-\frac{r_1^2}{R^2}\left(z-\frac{z}{R}+\frac{z^2}{R^2}-\frac{z^3}{R^3}+\ldots\right)\right]-v\frac{6z^2}{h^3}M\right\}. \tag{15}$$

In the classical theory of bending of thin shells, the term z/R and all its higher order terms are neglected. In the present formulation, the term z/R is retained but all its higher order terms $(z/R)^2$, $(z/R)^3$, etc. are neglected. The resulting expression for $w(x, \phi, z)$ is now given by eqn (3).

In order to obtain consistent assumptions for the displacements $u(x, \phi, z)$ and $v(x, \phi, z)$, the following strain–displacement relations are used:

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\gamma_{xz}=\frac{\tau_{xz}}{G} \tag{16}$$

$$\frac{\partial v}{\partial z}-\frac{v}{r}+\frac{\partial w}{r\,\partial\phi}=\gamma_{\phi z}=\frac{\tau_{\phi z}}{G}. \tag{17}$$

Substituting for the appropriate shearing stress from expressions (8) and (9) into eqns (16) and (17) and integrating both expressions with respect to z, we obtain the originally postulated expressions (1) and (2) for the u and v displacements, respectively.

In the shell theory that follows the distributed loads p_{xi}, p_{xo}, $p_{\phi i}$ and $p_{\phi o}$ are omitted for simplicity and conciseness. The reader may choose to include them by following the procedure outlined below.

Stress components σ_x, σ_ϕ, $\tau_{x\phi}$

In order to obtain the σ_x, σ_ϕ, $\tau_{x\phi}$ stresses, use is made of the following three stress–strain relations

$$\sigma_x=\frac{E}{1-v^2}[\varepsilon_x+v\varepsilon_\phi]+\frac{v}{1-v}\sigma_z \tag{18}$$

$$\sigma_\phi=\frac{E}{1-v^2}[\varepsilon_\phi+v\varepsilon_x]+\frac{v}{1-v}\sigma_z \tag{19}$$

$$\tau_{x\phi}=G\gamma_{x\phi} \tag{20}$$

together with the following strain–displacement relations

$$\varepsilon_x=\frac{\partial u}{\partial x} \tag{21}$$

$$\varepsilon_\phi=\frac{\partial v}{r\,\partial\phi}+\frac{w}{r}=\frac{1}{R+z}\left(\frac{\partial v}{\partial\phi}+w\right)=\frac{1}{1+z/R}\left(\frac{\partial v}{R\,\partial\phi}+\frac{w}{R}\right) \tag{22}$$

$$\gamma_{x\phi} = \frac{\partial u}{r\partial\phi} + \frac{\partial v}{\partial x} = \frac{1}{1+z/R}\frac{\partial u}{R\partial\phi} + \frac{\partial v}{\partial x}. \tag{23}$$

Substituting for the displacements u and v from eqns (1) and (2), respectively, into expressions (21), (22) and (23) and substituting the resulting strain expressions into eqns (18), (19) and (20), we obtain the following expressions for the stresses:

$$\begin{aligned}\sigma_x = \frac{E}{1-v^2}\Bigg[&\frac{\partial u_0}{\partial x} - \frac{\partial^2 w_0}{\partial x^2}z + \frac{2v}{Eh^3}\frac{\partial^2 M}{\partial x^2}z^3 + \frac{1}{2Gh}\frac{\partial Q_x}{\partial x}z\left[\left(3-\frac{4z^2}{h^2}\right) - \frac{3z}{R}\left(\frac{1}{2}-\frac{z^2}{h^2}\right)\right]\\
&- \frac{1}{Ec_1}\frac{\partial^2 p_i}{\partial x^2}\left[\frac{z^2}{2} - \frac{r_2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right] - \frac{1}{Ec_2}\frac{\partial^2 p_o}{\partial x^2}\left[\frac{z^2}{2} - \frac{r_1}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right]\\
&+ v\Bigg\{\frac{\partial v_0}{R\partial\phi} - \frac{\partial^2 w_0}{R^2\partial\phi^2}\left(z-\frac{z^2}{R}\right) + \frac{2v}{Eh^3}\frac{\partial^2 M}{R^2\partial\phi^2}z^3\left(1-\frac{3z}{4R}\right)\\
&+ \frac{\partial Q_\phi}{R\partial\phi}z\left[\left(3-\frac{4z^2}{h^2}\right) - \left(\frac{1}{2}-\frac{z^2}{h^2}\right)\right] - \frac{1}{Ec_1}\frac{\partial^2 p_i}{R^2\partial\phi^2}\left[\frac{z^2}{2} - \frac{r_2^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right.\\
&\left.- \frac{2}{R}\left(\frac{z^3}{3} - \frac{r_2^2}{R^2}\left(\frac{z^3}{3}-\frac{z^4}{4R}\right)\right)\right] - \frac{1}{Ec_2}\frac{\partial^2 p_o}{R^2\partial\phi^2}\left[\frac{z^2}{2} - \frac{r_1^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right.\\
&\left.- \frac{2}{R}\left(\frac{z^3}{3} - \frac{r_1^2}{R^2}\left(\frac{z^3}{3}-\frac{z^4}{4R}\right)\right)\right] + \frac{1}{R+z}\left[w_0 + \frac{p_i}{Ec_1}\left(z - \frac{r_2^2}{R^2}\left(z-\frac{z^2}{R}\right)\right)\right.\\
&\left.\left.+ \frac{p_o}{Ec_2}\left(z - \frac{r_1^2}{R^2}\left(z-\frac{z^2}{R}\right)\right) - v\frac{6z^2}{Eh^3}\right]\Bigg\}\Bigg] + \frac{v}{1-v}\Bigg\{\frac{p_i}{c_1}\left[1 - \frac{r_2^2}{R^2}\left(1-\frac{2z}{R}\right)\right]\\
&+ \frac{p_o}{c_2}\left[1 - \frac{r_1^2}{R^2}\left(1-\frac{2z}{R}\right)\right]\Bigg\}\end{aligned} \tag{24}$$

$$\begin{aligned}\sigma_\phi = \frac{E}{1-v^2}\Bigg[&\frac{\partial v_0}{R\partial\phi} - \frac{\partial^2 w_0}{R^2\partial\phi^2}\left(z-\frac{z^2}{R}\right) + \frac{1}{2Gh}\frac{\partial Q_\phi}{R\partial\phi}z\left[3-\frac{4z^2}{h^2} - \frac{3E}{R}\left(\frac{1}{2}-\frac{z^2}{h^2}\right)\right]\\
&+ \frac{2v}{Eh^3}\frac{\partial^2 M}{R^2\partial\phi^2}z^3\left(1-\frac{3z}{4R}\right)^2 - \frac{1}{Ec_1}\frac{\partial p_i^2}{R^2\partial\phi^2}\left[\frac{z^2}{2} - \frac{r_2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right.\\
&\left.- \frac{2}{R}\left(\frac{z^3}{3} - \frac{r_2^2}{3}\left(\frac{z^3}{3}-\frac{z^4}{4R}\right)\right)\right] - \frac{1}{Ec_2}\frac{\partial^2 p_o}{R^2\partial\phi^2}\left[\frac{z^2}{2} - \frac{r_1^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right.\\
&\left.- \frac{2}{R}\left(\frac{z^3}{3} - \frac{r_1^2}{R^2}\left(\frac{z^3}{3}-\frac{z^4}{4R}\right)\right)\right] + \frac{1}{R+z}\Bigg\{w_0 + \frac{p_i}{Ec_1}\left[z - \frac{r_2^2}{R^2}\left(z-\frac{z^2}{R}\right)\right]\\
&+ \frac{p_o}{Ec_2}\left[z - \frac{r_1^2}{R^2}\left(z-\frac{z^2}{R}\right)\right] - \frac{6vz^2}{Eh^3}M\Bigg\} + v\Bigg\{\frac{\partial u_0}{\partial x} - \frac{\partial^2 w_0}{\partial x^2}z + \frac{2v}{Eh^3}\frac{\partial^2 M}{\partial x^2}z^3\\
&+ \frac{1}{2Gh}\frac{\partial Q_x}{\partial x}z\left[3-\frac{4z^2}{h^2} - \frac{3z}{R}\left(\frac{1}{2}-\frac{z^2}{h^2}\right)\right] - \frac{1}{Ec_1}\frac{\partial^2 p_i}{\partial x^2}\left[\frac{z^2}{2}\frac{r^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right]\\
&- \frac{1}{Ec_2}\frac{\partial^2 p_o}{\partial x^2}\left[\frac{z^2}{2} - \frac{r_1^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right]\Bigg\}\Bigg] + \frac{v}{1-v}\Bigg\{\frac{p_i}{c_1}\left[1 - \frac{r_2^2}{R^2}\left(1-\frac{2z}{R}\right)\right]\\
&+ \frac{p_o}{c_2}\left[1 - \frac{r_1^2}{R^2}\left(1-\frac{2z}{R}\right)\right]\Bigg\}\end{aligned} \tag{25}$$

and

$$\tau_{x\phi} = \frac{E}{2(1+\nu)}\left\{\left(1+\frac{z}{R}\right)\frac{\partial v_0}{\partial x} + \frac{1}{1+z/R}\left[\frac{\partial u_0}{R\,\partial\phi} - 2\,\frac{\partial^2 w_0}{R\,\partial x\,\partial\phi}z\left(1+\frac{z}{2R}\right)\right]\right.$$
$$+\frac{4\nu}{Eh^3}\frac{\partial^2 M}{R\,\partial x\,\partial\phi}z^3 - \frac{3\nu}{2Eh^3}\frac{\partial^2 M}{R\,\partial x\,\partial\phi}\left(\frac{z^4}{R}\right)\left(1+\frac{z}{R}\right) + \frac{1}{2Gh}\left[\left(1-\frac{z}{R}\right)\frac{\partial Q_x}{R\,\partial\phi}\right.$$
$$\left.+\left(1+\frac{z}{R}\right)\frac{\partial Q_\phi}{\partial x}\right]z\left[\left(3-\frac{4z^2}{h^2}\right) - \frac{3z}{R}\left(\frac{1}{2}-\frac{z^2}{h^2}\right)\right] - \frac{2}{Ec_1}\frac{\partial^2 p_i}{R\,\partial x\,\partial\phi}\left[\frac{z^2}{2} - \frac{r_2^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{3R}\right)\right.$$
$$\left.-\frac{1}{R}\left(1+\frac{z}{R}\right)\left(\frac{z^3}{3} - \frac{r_2^2}{R^2}\left(\frac{z^3}{3}-\frac{z^4}{4R}\right)\right)\right] - \frac{2}{Ec_2}\frac{\partial^2 p_o}{R\,\partial x\,\partial\phi}\left[\frac{z^2}{2} - \frac{r_1^2}{R^2}\left(\frac{z^2}{2}-\frac{z^3}{ER}\right)\right.$$
$$\left.\left.-\frac{1}{R}\left(1+\frac{z}{R}\right)\left(\frac{z^3}{3} - \frac{r_1^2}{R^2}\left(\frac{z^3}{3}-\frac{r^4}{4R}\right)\right)\right]\right\}. \tag{26}$$

Stress couples and stress resultants on the middle surface

Making use of the definitions for the stress couples:

$$M_x = -\int_{-h/2}^{h/2} \sigma_x z\left(1+\frac{z}{R}\right)\mathrm{d}z \tag{27}$$

$$M_\phi = -\int_{h/2}^{h/2} \sigma_\phi z\,\mathrm{d}z \tag{28}$$

$$M_{x\phi} = -\int_{h/2}^{h/2} \tau_{x\phi} z\left(1+\frac{z}{R}\right)\mathrm{d}z \tag{29}$$

$$M_{\phi x} = -\int_{-h/2}^{h/2} \tau_{\phi x} z\,\mathrm{d}z, \tag{30}$$

we now substitute the expressions for the stresses from eqns (24), (25) and (26) into the respective relations for the stress couples to obtain:

$$M_x = D\left[\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{R^2\,\partial\phi^2} - \frac{\partial u_0}{R\,\partial x} - \nu\frac{\partial v_0}{R^2\,\partial\phi}\right] - D\frac{3\nu}{10Eh}\left(\frac{\partial^2 M}{\partial x^2} + \nu\frac{\partial^2 M}{R^2\,\partial\phi^2}\right)$$
$$-\frac{(1+\nu)h^2}{5(1-\nu^2)}\left(\frac{\partial Q_x}{\partial x} + \nu\frac{\partial Q_\phi}{R\,\partial\phi}\right) + \frac{Dh^2}{ER^3}\frac{\partial^2}{\partial x^2}\left\{\frac{p_i}{c_1}\left[\frac{r_2^2}{20} + \frac{1}{24}(R^2-r_2^2)\right]\right.$$
$$\left.+\frac{p_o}{c_2}\left[\frac{r_1^2}{20} + \frac{1}{24}(R^2-r_1^2)\right]\right\} + \nu\frac{Dh^2}{ER^3}\frac{\partial^2}{R^2\,\partial\phi^2}\left\{\frac{p_i}{c_1}\left[\frac{r_2^2}{20} + \frac{1}{24}(R^2-r_2^2)\right]\right.$$
$$\left.+\frac{p_o}{c_2}\left[\frac{r_1^2}{20} + \frac{1}{24}(R^2-r_1^2)\right]\right\} - \nu\frac{D}{ER^3}\left\{(2+\nu)\left[\frac{p_i}{c_1}(R^2-r_2^2) + \frac{p_o}{c_2}(R^2-r_1^2)\right]\right.$$
$$\left.+2(1+\nu)\left[\frac{p_i}{c_1}r_2^2 + \frac{p_o}{c_2}r_1^2\right]\right\} + \frac{Dh^2}{30ER^3}\left[\frac{R^2-r_2^2}{c_1}\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{R^2\,\partial\phi^2}\right)p_i\right.$$
$$\left.+\frac{R^2-r_1^2}{c_2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{R^2\,\partial\phi^2}\right)p_o\right] \tag{31}$$

$$M_{\phi} = D\left[\frac{\partial^2 w_0}{R^2 \partial \phi^2} + \frac{w_0}{R^2} + \nu \frac{\partial^2 w_0}{\partial x^2}\right] - D\frac{3\nu}{10hE}\left(\frac{M}{R^2} + \frac{\partial^2 M}{R^2 \partial \phi^2} + \nu \frac{\partial^2 M}{\partial x^2}\right)$$
$$-\frac{(1+\nu)h^2}{5(1-\nu^2)}\left(\frac{\partial Q_{\phi}}{R\partial \phi} + \nu \frac{\partial Q_x}{\partial x}\right) + \frac{Dh^2}{20ER^3}\left[\left(1 + \frac{\partial^2}{R^2 \partial \phi^2} + \nu \frac{\partial^2}{\partial x^2}\right)\left(\frac{p_i}{c_1} r_2^2 + \frac{p_o}{c_2} r_1^2\right)\right]$$
$$-\nu \frac{D}{ER^3}\left[\frac{p_i}{c_1}(R^2 - r_2^2) + \frac{p_o}{c_2}(R^2 - r_1^2) + 2(1+\nu)\left(\frac{p_i}{c_1} r_2^2 + \frac{p_o}{c_2} r_1^2\right)\right] \quad (32)$$

$$M_{x\phi} = D(1-\nu)\left(\frac{\partial^2 w_0}{R\partial x \partial \phi} - \frac{\partial v_0}{R\partial x} - \frac{3\nu}{10hE}\frac{\partial^2 M}{R\partial x \partial \phi}\right) - (1-\nu)\left(\frac{\partial Q_x}{R\partial \phi} + \frac{\partial Q_{\phi}}{\partial x}\right)\frac{h^2}{10}$$
$$+(1-\nu)\frac{Dh^2}{ER^3}\frac{\partial^2}{R\partial x \partial \phi}\left\{\frac{p_i}{c_1}\left[\frac{r_2^2}{20} + \frac{1}{24}(R^2 - r_2^2)\right] + \frac{p_o}{c_2}\left[\frac{r_1^2}{20} + \frac{1}{24}(R^2 - r_1^2)\right]\right\}$$
$$+(1-\nu)\frac{Dh^2}{30ER^3}\frac{\partial^2}{R\partial x \partial \phi}\left[\frac{p_i}{c_1}(R^2 - r_2^2) + \frac{p_o}{c_2}(R^2 - r_1^2)\right] \quad (33)$$

$$M_{\phi x} = D(1-\nu)\left(\frac{\partial^2 w_0}{R\partial x \partial \phi} - \frac{3\nu}{10hE}\frac{\partial^2 M}{R\partial x \partial \phi} + \frac{1}{2R}\frac{\partial u_0}{R\partial \phi} - \frac{1}{2R}\frac{\partial v_0}{\partial x}\right)$$
$$-(1-\nu)\frac{h^2}{10}\left(\frac{\partial Q_x}{R\partial \phi} + \frac{\partial Q_{\phi}}{\partial x}\right) + (1-\nu)\frac{Dh^2}{ER^3}\frac{\partial^2}{R\partial x \partial \phi}\left(\frac{p_i}{c_1}\frac{r_2^2}{20} + \frac{p_o}{c_2}\frac{r_1^2}{20}\right). \quad (34)$$

In eqns (31) and (33), the following terms at the end of those expressions

$$\frac{Dh^2}{30ER^3}\left[\frac{R^2 - r_2^2}{c_1}\left(\frac{\partial^2}{\partial x^2} + \nu \frac{\partial^2}{R^2 \partial \phi^2}\right)p_i + \frac{R^2 - r_1^2}{c_2}\left(\frac{\partial^2}{\partial x^2} + \nu \frac{\partial^2}{R^2 \partial \phi^2}\right)p_o\right]$$

and

$$(1-\nu)\frac{Dh^2}{30ER^3}\frac{\partial^2}{R\partial x \partial \phi}\left[\frac{p_i}{c_1}(R^2 - r_2^2) + \frac{p_o}{c_2}(R^2 - r_1^2)\right],$$

respectively, may be ignored. These terms are of negligible magnitude when compared to the remaining terms of these expressions. This can also be shown in the case of the average transverse displacement $\bar{w}$ discussed later whereby these terms are of negligible magnitude when compared to the remaining terms in that expression. In the remaining formulation similar terms will be ignored for the same reasons.

Substituting for the stresses σ_x, σ_{ϕ} and $\tau_{x\phi}$ from eqns (24)–(26), respectively, into the following definitions for the stress resultants

$$N_x = \int_{-h/2}^{h/2} \sigma_x \left(1 + \frac{z}{R}\right) dz \quad (35)$$

$$N_{\phi} = \int_{-h/2}^{h/2} \sigma_{\phi}\, dz \quad (36)$$

$$N_{x\phi} = \int_{-h/2}^{h/2} \tau_{x\phi}\left(1 + \frac{z}{R}\right) dz \quad (37)$$

$$N_{\phi x} = \int_{-h/2}^{h/2} \tau_{\phi x}\, dz, \quad (38)$$

we obtain the following expressions for the stress resultants:

$$N_x = \frac{Eh}{1-\nu^2}\left\{\frac{\partial u_0}{\partial x} + \nu\frac{w_0}{R} + \nu\frac{\partial v_0}{R\partial\phi} - \frac{3\nu^2 M}{10REh} + \frac{\nu h^2}{20ER^4}\left(\frac{p_i}{c_1}r_1^2 + \frac{p_o}{c_2}r_2^2\right)\right.$$
$$\left. - \frac{h^2}{20ER^2}\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{R^2\partial\phi^2}\right)\left[\frac{p_i}{c_1}(R^2-r_2^2) + \frac{p_o}{c_2}(R^2-r_1^2)\right]\right\}$$
$$- \frac{D}{R}\left[\frac{\partial^2 w_0}{\partial x^2} - \frac{3\nu}{10hE}\frac{\partial^2 M}{\partial x^2} + \frac{h^2}{20ER^3}\frac{\partial^2}{\partial x^2}\left(\frac{p_i}{c_1}r_2^2 + \frac{p_o}{c_2}r_1^2\right)\right]$$
$$+ \frac{\nu}{1-\nu}\frac{h}{R^2}\left[\frac{p_i}{c_1}(R^2-r_2^2) + \frac{p_o}{c_2}(R^2-r_1^2)\right]. \tag{39}$$

$$N_\phi = \frac{Eh}{1-\nu^2}\left\{\frac{\partial v_0}{R\partial\phi} + \nu\frac{\partial u_0}{\partial x} - \frac{h^2}{24ER^2}\left(\frac{\partial^2}{R^2\partial\phi^2} + \nu\frac{\partial^2}{\partial x^2}\right)\left[\frac{p_i}{c_1}(R^2-r_2^2)\right.\right.$$
$$\left.\left. + \frac{p_o}{c_2}(R^2-r_1^2)\right] + \frac{1}{R}\left[w_0 - \frac{3\nu}{10Eh}M + \frac{h^2}{20ER^3}\left(\frac{p_i}{c_1}r_2^2 + \frac{p_o}{c_2}r_1^2\right)\right]\right\}$$
$$+ \frac{D}{R}\left\{\left(\frac{\partial^2}{R^2\partial\phi^2} + \frac{1}{R^2}\right)\left[w_0 - \frac{3\nu}{10Eh}M + \frac{h^2}{20ER^3}\left(\frac{p_i}{c_1}r_2^2 + \frac{p_o}{c_2}r_1^2\right)\right]\right.$$
$$\left. - \frac{(1+\nu)h}{10(1-\nu 2)}\left(\frac{\partial Q_x}{R\partial\phi} + \frac{\partial Q_\phi}{\partial x}\right)\right\} + \frac{\nu}{1-\nu}\frac{h}{R^2}\left[\frac{p_i}{c_1}(R^2-r_2^2) + \frac{p_o}{c_2}(R^2-r_1^2)\right]. \tag{40}$$

$$N_{x\phi} = \frac{Eh}{1-\nu^2}\left(\frac{1-\nu}{2}\right)\left\{\frac{\partial u_0}{R\partial\phi} + \frac{\partial v_0}{\partial x} - \frac{h^2}{12ER^2}\frac{\partial^2}{R\partial x\partial\phi}\left[\frac{p_i}{c_1}(R^2-r_2^2)\right.\right.$$
$$\left.\left. + \frac{p_o}{c_2}(R^2-r_1^2)\right]\right\} - \frac{D}{R}\left(\frac{1-\nu}{2}\right)\left\{\frac{\partial^2 w_0}{R\partial x\partial\phi} - \frac{3}{10Eh}\frac{\partial^2 M}{R\partial x\partial\phi}\right.$$
$$+ \frac{h^2}{10ER^3}\frac{\partial^2}{R\partial x\partial\phi}\left(\frac{p_i}{c_1}r_2^2 + \frac{p_o}{c_2}r_1^2\right) - \frac{(1+\nu)h^2}{10(1-\nu^2)}\left(\frac{\partial Q_x}{R\partial\phi} + \frac{\partial Q_\phi}{\partial x}\right)$$
$$\left. - \frac{1}{R}\frac{\partial}{\partial x}\left[v_0 - \frac{1}{20ER^3}\frac{\partial}{R\partial\phi}\left(\frac{p_i}{c_1}(R^2-r_2^2) + \frac{p_o}{c_2}(R^2-r_1^2)\right)\right]\right\}, \tag{41}$$

and

$$N_{\phi x} = \frac{Eh}{1-\nu^2}\left(\frac{1-\nu}{2}\right)\left\{\frac{\partial u_0}{R\partial\phi} + \frac{\partial v_0}{\partial x} - \frac{h^2}{12ER^2}\frac{\partial^2}{R\partial x\partial\phi}\left[\frac{p_i}{c_1}(R^2-r_2^2)\right.\right.$$
$$\left.\left. + \frac{p_o}{c_2}(R^2-r_1^2)\right]\right\} + \frac{D}{R}\left(\frac{1-\nu}{2}\right)\left\{\frac{\partial^2}{R\partial x\partial\phi}\left[w_0 - \frac{3}{10Eh}M\right.\right.$$
$$\left. + \frac{h^2}{10ER^3}\left(\frac{p_i}{c_1}r_2^2 + \frac{p_o}{c_2}r_1^2\right)\right] - \frac{(1-\nu)h^2}{10(1-\nu^2)}\left(\frac{\partial Q_x}{R\partial\phi} + \frac{\partial Q_\phi}{\partial x}\right)$$
$$\left. + \frac{1}{R}\frac{\partial}{R\partial\phi}\left[u_0 - \frac{h^2}{20ER^2}\frac{\partial}{\partial x}\left(\frac{p_i}{c_1}(R^2-r_2^2) + \frac{p_o}{c_2}(R^2-r_1^2)\right)\right]\right\}. \tag{42}$$

Average displacements $\bar{w}$, $\bar{u}$, $\bar{v}$ and rotations ϕ_x, ϕ_ϕ

For identifying the proper boundary conditions of the derived shell theory, average displacements $\bar{w}$, $\bar{u}$ and $\bar{v}$, and average rotations ϕ_x and ϕ_ϕ are introduced. The rotations

ϕ_x and ϕ_ϕ are for sections x = constant, and ϕ = constant, respectively. The average transverse displacement $\bar{w}$ is obtained by equating the work of the transverse shear stress $\tau_{\phi z}$ due to displacement w to the work of the transverse shear resultant Q_ϕ due to average displacement $\bar{w}$ (see Voyiadjis and Baluch, 1981):

$$\int_{-h/2}^{h/2} \tau_{\phi z} w \, dz = Q_\phi \bar{w}. \tag{43}$$

The resulting expression for $\bar{w}$ is given by

$$\bar{w} = w_0 - \frac{3v}{10hE} M + \frac{h^2}{20ER^3}\left(\frac{p_i}{c_1} r_2^2 + \frac{p_o}{c_2} r_1^2\right). \tag{44}$$

Similarly, in order to obtain $\bar{u}$, $\bar{v}$, ϕ_x, and ϕ_ϕ we use the following equations:

$$\int_{-h/2}^{h/2} \sigma_x u\left(1+\frac{z}{R}\right) dz = N_x \bar{u} + M_x \phi_x \tag{45}$$

and

$$\int_{-h/2}^{h/2} \sigma_\phi v \, dz = N_\phi \bar{v} + M_\phi \phi_\phi. \tag{46}$$

The resulting expressions for $\bar{u}$, $\bar{v}$, ϕ_x and ϕ_ϕ are given, respectively, by

$$\bar{u} = u_0 - \frac{h^2}{24ER^2}\left[\frac{1}{c_1}\frac{\partial p_i}{\partial x}(R^2 - r_2^2) + \frac{1}{c_2}\frac{\partial p_o}{\partial x}(R^2 - r_1^2)\right] \tag{47}$$

$$\bar{v} = v_0 - \frac{h^2}{24ER^2}\left[\frac{1}{c_1}\frac{\partial p_i}{R\partial\phi}(R^2 - r_2^2) + \frac{1}{c_2}\frac{\partial p_o}{R\partial\phi}(R^2 - r_1^2)\right] \tag{48}$$

$$\phi_x = \frac{\partial \bar{w}}{\partial x} - \frac{6}{5}\frac{Q_x}{Gh} \tag{49}$$

and

$$\phi_\phi = \frac{\partial \bar{w}}{R\partial\phi} - \frac{6}{5}\frac{Q_\phi}{Gh} - \frac{\bar{v}}{R}. \tag{50}$$

Let the shear angles γ_x and γ_ϕ be defined such that

$$\gamma_x = \frac{Q_x}{T} \tag{51}$$

and

$$\gamma_\phi = \frac{Q_\phi}{T} \tag{52}$$

where

$$T = \frac{5}{6} Gh. \tag{53}$$

We therefore obtain

$$\phi_x = \frac{\partial \bar{w}}{\partial x} - \gamma_x \tag{54}$$

and

$$\phi_\phi = \frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi - \frac{\bar{v}}{R}. \tag{55}$$

Equations (49), (50) and (53) indicate that the correction factor of the transverse shear deformation in the present refined shell theory is 5/6.

The stress resultants and stress couples may be expressed in a more concise manner in terms of $\bar{u}$, $\bar{v}$, $\bar{w}$, γ_x and γ_ϕ as follows:

$$M_x = D\left[\frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right) + \nu\frac{\partial}{R\,\partial \phi}\left(\frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi\right) - \frac{\partial \bar{u}}{R\,\partial x} - \nu\frac{\partial \bar{v}}{R^2\,\partial \phi}\right] + k_1 p_i + k_2 p_o \tag{56}$$

$$M_\phi = D\left[\frac{\partial}{R\,\partial \phi}\left(\frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi\right) + \frac{\bar{w}}{R^2} + \nu\frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right)\right] + k_3 p_i + k_4 p_o \tag{57}$$

$$M_{x\phi} = D\left(\frac{1-\nu}{2}\right)\left[\frac{\partial}{R\,\partial \phi}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right) + \frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi\right) - \frac{2}{R}\frac{\partial \bar{v}}{\partial x}\right] \tag{58}$$

$$M_{\phi x} = D\left(\frac{1-\nu}{2}\right)\left[\frac{\partial}{R\,\partial \phi}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right) + \frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi\right) + \frac{1}{R}\left(\frac{\partial \bar{u}}{R\,\partial \phi} - \frac{\partial \bar{v}}{\partial x}\right)\right] \tag{59}$$

$$N_x = \frac{Eh}{1-\nu^2}\left[\frac{\partial \bar{u}}{\partial x} + \nu\frac{\partial \bar{v}}{R\,\partial \phi} + \nu\frac{\bar{w}}{R}\right] - \frac{D}{R}\frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right) + k_5 p_i + k_6 p_o \tag{60}$$

$$N_\phi = \frac{Eh}{1-\nu^2}\left[\frac{\partial \bar{v}}{R\,\partial \phi} + \nu\frac{\partial \bar{u}}{\partial x} + \frac{\bar{w}}{R}\right] + \frac{D}{R}\left[\frac{\partial}{R\,\partial \phi}\left(\frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi\right) + \frac{\bar{w}}{R^2}\right] + k_5 p_i + k_6 p_o \tag{61}$$

$$N_{x\phi} = \frac{Eh}{1-\nu^2}\left(\frac{1-\nu}{2}\right)\left(\frac{\partial \bar{u}}{R\,\partial \phi} + \frac{\partial \bar{v}}{\partial x}\right) - \frac{D}{R}\left(\frac{1-\nu}{4}\right)\left[\frac{\partial}{R\,\partial \phi}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right) + \frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi\right) - \frac{2}{R}\frac{\partial \bar{v}}{\partial x}\right] \tag{62}$$

and

$$N_{\phi x} = \frac{Eh}{1-\nu^2}\left(\frac{1-\nu}{2}\right)\left(\frac{\partial \bar{u}}{R\,\partial \phi} + \frac{\partial \bar{v}}{\partial x}\right) + \frac{D}{R}\left(\frac{1-\nu}{4}\right)\left[\frac{\partial}{R\,\partial \phi}\left(\frac{\partial \bar{w}}{\partial x} - \gamma_x\right) + \frac{\partial}{\partial x}\left(\frac{\partial \bar{w}}{R\,\partial \phi} - \gamma_\phi\right) + \frac{2}{R}\frac{\partial \bar{u}}{R\,\partial \phi}\right] \tag{63}$$

where

$$k_1 = -\nu\frac{D}{ER^3}\frac{1}{c_1}[(2+\nu)(R^2 - r_1^2) + 2(1+\nu)r_1^2] \tag{64}$$

$$k_2 = -v\frac{D}{ER^3}\frac{1}{c_2}[(2+v)(R^2-r_1^2)+2(1+v)r_1^2] \tag{65}$$

$$k_3 = -v\frac{D}{ER^3}\frac{1}{c_1}[R^2-r_2^2+2(1+v)r_2^2] \tag{66}$$

$$k_4 = -v\frac{D}{ER^3}\frac{1}{c_2}[R^2-r_1^2+2(1+v)r_1^2] \tag{67}$$

$$k_5 = \frac{v}{1-v}\frac{h}{R^2}\frac{1}{c_1}(R^2-r_2^2) \tag{68}$$

$$k_6 = \frac{v}{1-v}\frac{h}{R^2}\frac{1}{c_2}(R^2-r_1^2). \tag{69}$$

The transverse shears Q_x and Q_ϕ are given by

$$Q_x = T\gamma_x \tag{70}$$

and

$$Q_\phi = T\gamma_\phi. \tag{71}$$

These resulting constitutive equations of shells reduce to those given by Flugge (1960) when the shear deformation and radial effects are neglected. In this case, the average displacements are replaced by the middle surface displacements. The transverse shear forces Q_x and Q_ϕ are obtained in this case from the equilibrium equations in terms of the stress couples.

An alternate set of expressions for the stress resultants and stress couples may be obtained in terms of the average displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ and corresponding rotations ϕ_x and ϕ_ϕ. These equations are given by the following relations:

$$M_x = D\left[\frac{\partial\phi'_x}{\partial x}+v\frac{\partial\phi_\phi}{R\,\partial\phi}\right]+k_1p_i+k_2p_o \tag{72}$$

$$M_\phi = D\left[\frac{\partial\phi_\phi}{R\,\partial\phi}+v\frac{\partial\phi'_x}{\partial x}+\frac{1}{R}\left(\frac{\bar{w}}{R}+v\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{R\,\partial\phi}\right)\right]+k_3p_i+k_4p_o \tag{73}$$

$$M_{x\phi} = D\left(\frac{1-v}{2}\right)\left[\frac{\partial\phi'_x}{R\,\partial\phi}+\frac{\partial\phi_\phi}{\partial x}+\frac{1}{R}\left(\frac{\partial\bar{u}}{R\,\partial\phi}-\frac{\partial\bar{v}}{\partial x}\right)\right] \tag{74}$$

$$M_{\phi x} = D\left(\frac{1-v}{2}\right)\left[\frac{\partial\phi'_x}{R\,\partial\phi}+\frac{\partial\phi_\phi}{\partial x}+\frac{2}{R}\frac{\partial\bar{u}}{R\,\partial\phi}\right] \tag{75}$$

$$N_x = \left(\frac{Eh}{1-v^2}\right)\left(\frac{\partial\bar{u}}{\partial x}+v\frac{\partial\bar{v}}{R\,\partial\phi}+v\frac{\bar{w}}{R}\right)-\frac{D}{R}\left(\frac{\partial\phi'_x}{\partial x}+\frac{1}{R}\frac{\partial\bar{u}}{\partial x}\right)+k_5p_i+k_6p_o \tag{76}$$

$$N_\phi = \left(\frac{Eh}{1-v^2}\right)\left(\frac{\partial v}{R\,\partial\phi}+\frac{\bar{w}}{R}+v\frac{\partial\bar{u}}{\partial x}\right)+\frac{D}{R}\left(\frac{\partial\phi_\phi}{R\,\partial\phi}+\frac{1}{R}\frac{\partial\bar{v}}{R\,\partial\phi}+\frac{\bar{w}}{R^2}\right)+k_5p_i+k_6p_o \tag{77}$$

$$N_{x\phi} = \left(\frac{Eh}{1-v^2}\right)\left(\frac{\partial\bar{u}}{R\,\partial\phi}+\frac{\partial\bar{v}}{\partial x}\right)\left(\frac{1-v}{2}\right)-\frac{D}{R}\left(\frac{1-v}{4}\right)\left[\frac{\partial\phi'_x}{R\,\partial\phi}+\frac{\partial\phi_\phi}{\partial x}+\frac{1}{R}\left(\frac{\partial\bar{u}}{R\,\partial\phi}-\frac{\partial\bar{v}}{\partial x}\right)\right] \tag{78}$$

and

$$N_{\phi x} = \left(\frac{Eh}{1-\nu^2}\right)\left(\frac{\partial \bar{u}}{R\partial \phi} + \frac{\partial \bar{v}}{\partial x}\right)\left(\frac{1-\nu}{2}\right) - \frac{D}{R}\left(\frac{1-\nu}{4}\right)\left[\frac{\partial \phi'_x}{R\partial \phi} + \frac{\partial \phi_\phi}{\partial x} + \frac{1}{R}\left(3\frac{\partial \bar{u}}{R\partial \phi} + \frac{\partial \bar{v}}{\partial x}\right)\right]. \tag{79}$$

The corresponding transverse shears are expressed by the following equations:

$$Q_x = T\left(\frac{\partial \bar{w}}{\partial x} - \phi'_x - \frac{\bar{u}}{R}\right) \tag{80}$$

$$Q_\phi = T\left(\frac{\partial \bar{w}}{R\partial \phi} - \phi_\phi - \frac{\bar{v}}{R}\right) \tag{81}$$

where ϕ'_x is defined by subtracting the term $\bar{u}/R$ from eqn (49) such that

$$\phi'_x = \phi_x - \bar{u}/R. \tag{49a}$$

Equilibrium equations and boundary conditions

For the case of small deformation analysis, the shell equilibrium equations are given below (Flugge, 1960):

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{\phi x}}{R\partial \phi} + p_x = 0 \tag{82}$$

$$\frac{\partial N_\phi}{R\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} + \frac{Q_\phi}{R} + p_\phi = 0 \tag{83}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{R\partial \phi} - \frac{N_\phi}{R} + p_z = 0 \tag{84}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{\phi x}}{R\partial \phi} + Q_x + m_x = 0 \tag{85}$$

$$\frac{\partial M_\phi}{R\partial \phi} + \frac{\partial M_{x\phi}}{\partial x} + Q_\phi + m_\phi = 0 \tag{86}$$

$$N_{x\phi} - N_{\phi x} + \frac{M_{\phi x}}{R} = 0. \tag{87}$$

In the above equilibrium expressions p_x, p_ϕ and p_z are the equivalent distributed loads acting on the middle surface of the shell. For example, the load p_z is given by

$$p_z = p_i\left(1 - \frac{h}{2R}\right) + p_o\left(1 + \frac{h}{2R}\right). \tag{88}$$

m_x and m_ϕ are the equivalent distributed moments acting on the middle surface due to the distributed loads p_{xi}, p_{xo}, $p_{\phi i}$ and $p_{\phi o}$ acting on the surfaces of the shell. The sixth equation given above by expression (87) is identically satisfied consequently reducing the number of equilibrium equations to five. The stress resultants and couples may be expressed in terms of either $\bar{u}$, $\bar{v}$, $\bar{w}$, γ_x, and γ_ϕ or $\bar{u}$, $\bar{v}$, $\bar{w}$, ϕ_x, and ϕ_ϕ. We therefore have five unknowns to solve for, from the five remaining equilibrium equations (82)–(86).

The static and kinematic boundary conditions for this refined shell theory may be expressed in terms of either the quantities $(\bar{u}, \bar{v}, \bar{w}, \gamma_x, \gamma_\phi)$ or $(\bar{u}, \bar{v}, \bar{w}, \phi_x, \phi_\phi)$ together with the use of constitutive equations (57)–(71) or (72)–(81). There will be three boundary conditions for bending problems for each edge. The usual expressions for simple, fixed and free edges will be used to express these boundary expressions (see Niordson, 1985).

THE NONLINEAR NATURE OF THE STRESS DISTRIBUTION

The resulting nonlinear distribution through the thickness for the in-plane stresses in the proposed thick shell theory is due to the incorporation of the initial curvature of the shell, and the three-dimensional constitutive equations as obtained from relations (18)–(20). This effect becomes highly pronounced in thick shells by changing the magnitude of the maximum stress significantly as compared to the linear stress variation theory.

In the expressions of the in-plane stress components σ_x, σ_ϕ and $\tau_{x\phi}$ as given by eqns (24)–(26), respectively, nonlinear terms such as $1/(R+z)$ and z^2/R are involved. Consequently, the stresses given by the present theory have a nonlinear distribution along the thickness of the shell. Let us consider the simple case of a constant normal pressure and investigate the corresponding stress distribution of σ_ϕ through the thickness. In this case we have

$$\sigma_\phi = \frac{E}{1-\nu^2}\Bigg\{\left(\frac{\partial v_0}{R\,\partial\phi} + \frac{w_0}{R+z} + \nu\frac{\partial u_0}{\partial x}\right) - \left[\frac{\partial^2 w_0}{R^2\,\partial\phi^2}\left(1-\frac{z}{R}\right) + \nu\frac{\partial^2 w_0}{\partial x^2}\right]z$$
$$+ \frac{2\nu}{Eh}\left[\frac{\partial^2 M}{R^2\,\partial\phi^2}\left(1-\frac{3z}{4R}\right) + \nu\frac{\partial^2 M}{\partial x^2}\right]z^3 + \frac{1}{2Gh}\left[\frac{\partial Q_\phi}{R\,\partial\phi} + \nu\frac{\partial Q_x}{\partial x}\right]z$$
$$\cdot\left[3 - \frac{4z^2}{h^2} - \frac{3z}{R}\left(\frac{1}{2} - \frac{z^2}{h^2}\right)\right] + \frac{1}{R+z}\frac{1}{E}\left[\frac{p_i}{c_1}\left(z - \frac{r_2^2}{R^2}\left(z - \frac{z^2}{R}\right)\right)\right.$$
$$\left.\left. + \frac{p_o}{c_2}\left(z - \frac{r_1^2}{R^2}\left(z - \frac{z^2}{R}\right)\right) - \nu\frac{6z^2}{h^3}M\right]\right\} + \frac{\nu}{1-\nu}\left[\frac{p_i}{c_1}\left(1 - \frac{r_2^2}{(R+z)^2}\right)\right.$$
$$\left. + \frac{p_o}{c_2}\left(1 - \frac{r_1^2}{(R+z)^2}\right)\right]. \tag{89}$$

In eqn (89) all the terms are nonlinear in z except for the terms associated with $\partial v_0/R\,\partial\phi$, $\partial u_0/\partial x$ and $\partial^2 w_0/\partial x^2$.

A number of problems will be investigated in order to assess the accuracy of the distribution of σ_ϕ through the thickness. The stress distribution obtained using the presented theory will be compared with various theories.

Circular arch subjected to pure bending

Let us consider the problem of a circular arch of rectangular cross-section with unit width subjected to pure bending. In this case we have

$$p_i = p_o = Q_x = Q_\phi = \frac{\partial^2 M}{\partial x^2} = \frac{\partial^2 M}{R\,\partial\phi^2} = u = 0 \tag{90}$$

and therefore eqn (89) reduces to the following expression:

$$\sigma_\phi = E\left[\frac{\partial v_0}{R\,\partial\phi} + \frac{w_0}{R} - \frac{6\nu z}{REh^3}M_\phi\left(1 - \frac{z}{R} + \frac{z^2}{R^2}\right) - \left(\frac{\partial^2 w_0}{R^2\partial\phi^2} + \frac{w_0}{R^2}\right)\left(z - \frac{z^2}{R}\right)\right]. \tag{91}$$

By considering $M_\phi/E \ll w_0$, the above equation may be approximated to the following:

$$\sigma_\phi = E\left[\frac{\partial v_0}{R\,\partial\phi} + \frac{w_0}{R}\right] - \frac{M_\phi}{(h^3/12)}z\left(1 - \frac{z}{R}\right). \tag{92}$$

From the elasticity solution we obtain the circumferential stress at the middle surface of the arch to be $E(\partial v_0/R\,\partial\phi + w_0/R)$ and is independent of the angle ϕ. The proposed shell theory indicates also that the middle surface is not a zero stress surface (neutral surface). Equation (92) may be rewritten as

$$\sigma_\phi = -\frac{M_\phi}{(h^3/12)}z + \left(\sigma_\phi|_{z=0} + \frac{M_\phi}{(h^3/12)}\frac{z^2}{R}\right). \tag{93}$$

The two terms inside the parentheses in the above equation indicate the modifications made to the stress distribution of the elementary theory through the use of the proposed shell theory. We now express the maximum and minimum values of σ_ϕ as follows:

$$\sigma_\phi = \alpha\frac{M_\phi}{r_1^2} \tag{94}$$

where the values of α as a function of r_2/r_1 for different formulations (see Ugural and Fenster, 1975) are listed in Table 1. In this table, the maximum and minimum values of σ_ϕ are listed for three different values of the aspect ratio r_2/r_1, namely 1.5, 2.0 and 3.0. The corresponding aspect ratios for R/h are 2.5, 1.5 and 1, respectively. These ratios correspond to extremely thick cylinders. It is clear from Table 1 that the proposed theory gives good results even for the case of a circular arch with $R/h = 1$.

Thick cylinder subjected to uniform pressures

We now investigate the stress distribution of σ_ϕ for a thick cylinder subjected to uniform pressure p_i and p_o. In this case we have

$$v = Q_\phi = \frac{\partial M_\phi}{\partial\phi} = 0 \tag{95}$$

and

$$w = w(z). \tag{96}$$

The stress σ_ϕ using the proposed theory is expressed in this case as follows:

$$\sigma_\phi = \frac{E}{R+z}\left\{w_0 + \frac{p_i}{Ec_1}\left[z - \frac{r_2^2}{R^2}\left(z - \frac{z^2}{R}\right)\right] + \frac{p_o}{Ec_2}\left[z - \frac{r_1^2}{R^2}\left(z - \frac{z^2}{R}\right)\right]\right\}. \tag{97}$$

The corresponding exact elasticity solution for this problem is given by

$$\sigma_\phi^e = \left[1 + \frac{r_2^2}{(R+z)^2}\right]\frac{p_i}{c_1} + \left[1 + \frac{r_1^2}{(R+z)^2}\right]\frac{p_o}{c_2}. \tag{98}$$

Table 1

r_2/r_1	Elementary theory	Winkler's theory $r = r_1$	Winkler's theory $r = r_2$	Proposed shell theory $r = r_1$	Proposed shell theory $r = r_2$	Elasticity solution $r = r_1$	Elasticity solution $r = r_2$
1.5	±24	−26.971	20.607	−27.971	20.029	−27.858	21.275
2	±6	−7.725	4.863	−7.642	4.358	−7.755	4.917
3	±1.5	−2.285	1.095	−2.105	0.895	−2.292	1.130

From the elasticity theory, we have

$$w_0 = \frac{R}{E}\sigma_\phi|_{z=0} \tag{99}$$

where

$$\sigma_\phi|_{z=0} = \left[\left(1+\frac{r_2^2}{R^2}\right)\frac{p_i}{c_1}+\left(1+\frac{r_1^2}{R^2}\right)\frac{p_o}{c_2}\right]. \tag{100}$$

Substituting for w_0 from eqns (99) and (100) into expression (97), we obtain the following expression for σ_ϕ:

$$\sigma_\phi = \frac{R}{R+z}\left\{\left[1+\frac{r_2^2}{R^2}+\frac{z}{R}\left(1-\frac{r_2}{R(R+z)}\right)\right]\frac{p_i}{c_1}+\left[1+\frac{r_1^2}{R^2}+\frac{z}{R}\left(1-\frac{r_1}{R(R+z)}\right)\right]\frac{p_o}{c_2}\right\}. \tag{101}$$

It can be easily shown that σ_ϕ as obtained from eqn (101) of the proposed shell theory is identical to that of the exact elasticity solution expressed by eqn (98).

Gupta and Khatua (1978) in their derivation of a thick shell superparametric finite element proposed a modification in the expression for the circumferential stress σ_ϕ. Their modified expression is given by

$$\sigma_\phi = \frac{R}{R+z}\sigma_0 \tag{102}$$

where σ_0 is the average hoop stress. We note that Gupta and Khatua's scheme cannot distinguish the difference between the internal and external pressures. We also note that Winkler's theory (see Ugural and Fenster, 1975) is not valid for this case of loading.

EQUIVALENT FORMULATION FOR THICK PLATE THEORY

It is relatively simple to reduce the proposed shell theory to a thick plate theory. The coefficients $k_1, k_2, \ldots$, etc. reduce to the following:

$$k_1 = k_2 = k_3 = k_4 = -\frac{6\nu(1+\nu)}{5Eh}D \tag{103}$$

for

$$\frac{1}{r} = \frac{1}{R}\left(1-\frac{z}{R}+\frac{z^2}{R^2}-\frac{z^3}{R^3}+\ldots\right) \tag{104}$$

and

$$k_5 = k_6 = \frac{\nu h}{2(1-\nu)} \tag{105}$$

as R approaches infinity. In this case the stress resultants, and stress couples expressions reduce to:

$$N_x = \frac{Eh}{1-\nu^2}\left(\frac{\partial \bar{u}}{\partial x}+\nu\frac{\partial \bar{v}}{\partial y}\right)+k_5(p_i+p_o) \tag{106}$$

$$N_y = \frac{Eh}{1-\nu^2}\left(\frac{\partial \bar{v}}{\partial y} + \nu\frac{\partial \bar{u}}{\partial x}\right) + k_5(p_i + p_o) \quad (107)$$

$$N_{xy} = N_{yx} = Gh\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) \quad (108)$$

$$Q_x = T\left(\frac{\partial \bar{w}}{\partial x} - \phi_x\right) \quad (109)$$

$$Q_y = T\left(\frac{\partial \bar{w}}{\partial y} - \phi_y\right) \quad (110)$$

$$M_x = D\left(\frac{\partial \phi_x}{\partial x} + \nu\frac{\partial \phi_y}{\partial y}\right) + k_1(p_i + p_o) \quad (111)$$

$$M_y = D\left(\frac{\partial \phi_y}{\partial y} + \nu\frac{\partial \phi_x}{\partial x}\right) + k_1(p_i + p_o) \quad (112)$$

$$M_{xy} = M_{yx} = D\frac{1-\nu}{2}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right). \quad (113)$$

APPLICATION OF THE PROPOSED SHELL THEORY TO CIRCULAR ARCHES

Weak form of equilibrium equations

In the case of arches, we note that

$$\bar{u} = \phi_x = 0 \quad (114)$$

and the non-zero stress resultants and stress couples reduce to the following expressions:

$$N_\phi = S\left(\frac{\partial \bar{v}}{R\,\partial\phi} + \frac{\bar{w}}{R}\right) + \frac{D}{R}\left(\frac{\bar{w}}{R^2} + \frac{\partial \phi_\phi}{R\,\partial\phi} + \frac{1}{R}\frac{\partial \bar{v}}{R\,\partial\phi}\right) + k_5 p_i + k_6 p_o \quad (115)$$

$$Q_\phi = T\left(\frac{\partial \bar{w}}{R\,\partial\phi} - \phi_\phi - \frac{\bar{v}}{R}\right) \quad (116)$$

and

$$M_\phi = D\left(\frac{\partial \phi_\phi}{R\,\partial\phi} + \frac{1}{R}\frac{\partial \bar{v}}{R\,\partial\phi} + \frac{\bar{w}}{R^2}\right) + k_3 p_i + k_4 p_o \quad (117)$$

where $S = EA_0$ and A_0 is the cross-sectional area of the arch. The corresponding equations of equilibrium are given as follows:

$$\frac{\partial N_\phi}{R\,\partial\phi} + \frac{Q_\phi}{R} + p_\phi = 0 \quad (118)$$

$$\frac{\partial Q_\phi}{R\,\partial\phi} - \frac{N_\phi}{R} + p_z = 0 \quad (119)$$

$$\frac{\partial M_\phi}{R\,\partial\phi} + Q_\phi + m_\phi = 0. \quad (120)$$

Let $\delta\bar{v}$, $\delta\bar{w}$, and $\delta\phi_\phi$ be the test functions corresponding to $\bar{v}$, $\bar{w}$, and ϕ_ϕ, respectively.

The weak form of the equilibrium eqns (118)–(120) may now be expressed as

$$\int_L \left(\frac{\partial N_\phi}{R\,\partial\phi} + \frac{Q_\phi}{R} + p_\phi\right)\delta\bar{v}\,R\,\mathrm{d}\phi = 0 \tag{121}$$

$$\int_L \left(\frac{\partial Q_\phi}{R\,\partial\phi} - \frac{N_\phi}{R} + p_z\right)\delta\bar{w}\,R\,\mathrm{d}\phi = 0 \tag{122}$$

$$\int_L \left(\frac{\partial M_\phi}{R\,\partial\phi} + Q_\phi + m_\phi\right)\delta\phi\,R\,\mathrm{d}\phi = 0. \tag{123}$$

Substituting eqns (115)–(117) into the above integral equations and integrating by parts yields

$$R\int_L \left[\left(A\frac{\partial\delta\bar{v}}{\partial\phi}\frac{\partial\bar{v}}{\partial\phi} + \frac{T}{R^2}\bar{v}\delta\bar{v}\right) + \left(A\bar{w}\frac{\partial\delta\bar{v}}{\partial\phi} - \frac{T}{R^2}\frac{\partial\bar{w}}{\partial\phi}\delta\bar{v}\right) + \left(\frac{D}{R^3}\frac{\partial\delta\bar{v}}{\partial\phi}\frac{\partial\phi_\phi}{\partial\phi} + \frac{T}{R}\phi_\phi\delta\bar{v}\right)\right]\mathrm{d}\phi$$
$$= \int_L \delta\bar{v}p_\phi R\,\mathrm{d}\phi + (\mathrm{N}_\phi\delta\bar{v})|_{\phi_0}^{\phi_f}, \tag{124}$$

$$R\int_L \left[\left(A\delta\bar{w}\frac{\partial\bar{v}}{\partial\phi} - \frac{T}{R^2}\frac{\partial\delta\bar{w}}{\partial\phi}\bar{v}\right) + \left(\frac{T}{R^2}\frac{\partial\delta\bar{w}}{\partial\phi}\frac{\partial\bar{w}}{\partial\phi} + A\bar{w}\delta\bar{w}\right) + \left(\frac{D}{R^3}\delta\bar{w}\frac{\partial\phi_\phi}{\partial\phi} - \frac{T}{R}\frac{\partial\delta\bar{w}}{\partial\phi}\phi_\phi\right)\right]\mathrm{d}\phi$$
$$= \int_L R(p_z - k_5 p_t - k_6 p_o)\delta\bar{w}\,\mathrm{d}\phi + (Q_\phi\delta\bar{w})|_{\phi_0}^{\phi_f}, \tag{125}$$

and

$$R\int_L \left[\left(\frac{D}{R^3}\frac{\partial\delta\phi_\phi}{\partial\phi}\frac{\partial\bar{v}}{\partial\phi} + \frac{T}{R}\delta\phi_\phi\bar{v}\right) + \left(\frac{D}{R^3}\frac{\partial\delta\phi_\phi}{\partial\phi}\bar{w} - \frac{T}{R}\delta\phi_\phi\frac{\partial\bar{w}}{\partial\phi}\right) + \left(\frac{D}{R^2}\frac{\partial\delta\phi_\phi}{\partial\phi}\frac{\partial\phi_\phi}{\partial\phi}\right.\right.$$
$$\left.\left. + T\delta\phi_\phi\phi_\phi\right)\right]\mathrm{d}\phi = (M_\phi\delta\phi_\phi)|_{\phi_0}^{\phi_f} + R\int_L \left(m_\phi + k_3\frac{\partial p_t}{R\,\partial\phi} + k_4\frac{\partial p_o}{R\,\partial\phi}\right)\mathrm{d}\phi \tag{126}$$

where

$$A = \frac{S}{R^2} + \frac{D}{R^4}. \tag{127}$$

The left-hand side of eqns (124)–(126) result in the element stiffness matrix and the corresponding right-hand side of these equations give the external nodal load vector.

Finite element scheme

Since the emphasis here is to verify the accuracy of the proposed theory rather than to give an efficient finite element scheme, a simple finite element model is employed in this work.

The nodal variables for the circular arch are $\bar{v}$, $\bar{w}$ and ϕ_ϕ. For the weak form of the equilibrium equations a linear trial function for $\bar{v}$, $\bar{w}$ and ϕ_ϕ will be acceptable. Nevertheless, this gives poor performance and higher order trial functions need to be used. This is accomplished by employing interior degrees of freedom (bubble functions). The assumed trial functions are given by the following relations (Hu, 1981):

$$\bar{v}(\eta) = \alpha\bar{v}_i + \eta\bar{v}_j \tag{128}$$

$$\bar{w}(\eta) = \alpha \bar{w}_i + \eta \bar{w}_j + \alpha\eta\xi_1 + (\alpha^2\eta - \alpha\eta^2)\xi_2 \tag{129}$$

$$\phi_\phi(\eta) = \alpha\phi_i + \eta\phi_j + \alpha\eta\xi_3 \tag{130}$$

where

$$\eta = \frac{\phi}{\phi_L - \phi_0} \tag{131}$$

and

$$\alpha = 1 - \eta. \tag{132}$$

In the above equations ξ_1, ξ_2 *and* ξ_3 are the interior degrees of freedom in an element. Using the expression ξ_1, ξ_2 and ξ_3 for straight beams given by Hu (1981), we express $\bar{w}$ and ϕ_ϕ as follows:

$$w(\eta) = [\alpha + \lambda\alpha\eta(\alpha - \eta)]w_i + [\alpha\eta + \lambda\alpha\eta(\alpha - \eta)]\frac{L}{2}\phi_i + [\eta - \lambda\alpha\eta(\alpha - \eta)]w_j + [-\alpha\eta + \lambda\alpha\eta(\alpha - \eta)]\frac{L}{2}\phi_j \tag{133}$$

$$\phi(\eta) = -6\lambda\alpha\eta\frac{w_i}{L} + (\alpha - 3\lambda\alpha\eta)\phi_i + 6\lambda\alpha\eta\frac{w_j}{L} + (\eta - 3\lambda\alpha\eta)\phi_j \tag{134}$$

where

$$L = R(\phi_L - \phi_0) \tag{135}$$

and

$$\lambda = \frac{1}{\left(1 + 12\dfrac{D}{TL^2}\right)}. \tag{136}$$

In eqn (137) λ is the parameter for the shear deformation effect. For a slender beam, $\lambda \to 1$ and $\partial w/L\,\partial\eta \to \phi_\phi$ as $(h^2/L^2) \to 0$. These trial functions are valid for both thick and thin arches.

Numerical examples

The feasibility and accuracy of the proposed refined theory of thick shells presented in this paper are demonstrated by the following numerical examples on circular beams.

Example 1—*Cantilevered straight beam.* For the case when the radius R approaches infinity, we obtain a straight beam. In this example, the deflection of the free end of a cantilevered beam subjected to a concentrated load acting at the free end is investigated. This deflection is expressed as shown below

$$w_0 = \delta = \alpha\frac{PL^3}{EI}. \tag{137}$$

Different values of α are tabulated in Table 2 corresponding to different aspect ratios of (L/h). Full agreement is obtained between the presented theory and the exact elasticity solution for both thick and thin beams.

Example 2—*Thick circular cylinder.* In the discussion of the stress distribution, it is

Table 2. Deflection coefficient α

Type of solutions	L/h		
	5	10	100
2 Elements	0.3437	0.3359	0.3334
Exact	0.3437	0.3359	0.3333

Table 3. Transverse displacement w_0 (in.)

Type of solutions	R/h			
	3	5	10	100
Finite element solution	0.3220×10^{-4}	0.8745×10^{-4}	0.3422×10^{-3}	0.3343×10^{-1}
Exact solution	0.3272×10^{-4}	0.8840×10^{-4}	0.3442×10^{-3}	0.3345×10^{-1}

pointed out that the present theory gives the exact nonlinear stress distribution for a thick cylinder subjected to internal and/or external pressures provided the middle surface displacement w_0 can be accurately calculated. Based on the shell equations given in this paper, the transverse displacement of the middle surface w_0 using the finite element solution for various R/h ratios is listed in Table 3. The following data are utilized in order to obtain the solution:

Modulus of elasticity: $E = 3 \times 10^7$ psi
Poisson's ratio: $\nu = 0.3$
Thickness: $h = 10$ in.
Internal pressure: $p_i = 0$ psi
External pressure: $p_o = -10$ psi
(1 psi = 0.006895 MPa).

The finite element solution is in good agreement with the exact solution even for the case of extremely thick cylinders.

Example 3—Cantilevered circular arch. The case of a cantilevered circular arch subjected to an inward radial concentrated load at the free end is discussed here. The arch subtends an angle of $\pi/4$ radians. In Table 4, the results are listed for the deflection coefficient α as obtained by both the presented finite element formulation of the proposed thick shell theory and the exact solution. The coefficient α is obtained from the following equation:

$$w_0 = \delta = \alpha \frac{PR^3}{EI} \tag{138}$$

Good correspondence is obtained between the two formulations when eight elements are considered in the finite element mesh.

Table 4. Deflection coefficient α (for $R/h = 5$)

	Number of elements		
	4	8	Exact
α	0.1181	0.1440	0.1447

CONCLUSIONS

A two-dimensional theory for thick cylindrical shells is developed in this paper. By considering the shear strains, the transverse shear deformations are accounted for in the resulting shell equations. In the proposed theory, the initial curvature effect is incorporated in the stress distribution leading to an accurate nonlinear distribution of the in-plane stresses. Through the incorporation of the radial stresses to the proposed shell formulation, we obtain the resulting stress resultants and stress couples to be associated not only with the middle surface displacements of the shells, but also with the radial stresses explicitly. By using the constitutive equations of the three-dimensional theory of elasticity and incorporating the initial curvature effect on the stress resultants and couples, an accurate set of constitutive equations for two-dimensional shell theory is obtained.

The constitutive equations obtained here reduce to those given by Flugge (1960) when the shear deformations and the radial stress effects are neglected, while the average displacement is replaced by the middle surface displacements of the shell. The resulting proposed equations in this paper are slightly different when compared with the equations given by Sanders (1959), Koiter (1960) and Niordson (1978), primarily because they use the so-called effective stress resultant and stress couple tensors. These effective stresses are used in the variational derivation of the constitutive equations (see Niordson, 1985). However, even when both the shear deformation and radial stress effects are neglected, the stress distributions given in the present paper will still be nonlinear because the stresses are derived from the three-dimensional constitutive equations given by expressions (18)–(20).

The nonlinear distribution of in-plane stresses through the thickness of thick shells was ignored in the past and not accounted for in the shell theory formulation. This is not the case in the proposed formulation. This nonlinear distribution constitutes a very important ingredient for an accurate and reliable thick shell theory.

Similar to the shell theory of Sanders–Koiter, the presented shell equations are convenient for use in the finite element analysis. This is demonstrated here by the application of these equations to the circular arch analysis and by the authors in their forthcoming paper (Shi and Voyiadjis, 1990). The proposed theory is not only very useful in the analysis of thick shells, but also has the potential for use in the analysis of composite shells (see Noor and Burton, 1989). This theory is also important in applications of vibrations of shells where the shear deformation and stress distributions along the thickness direction play an important role.

Although only thick isotropic cylindrical shells are studied in this work, the methodology employed here may be extended to the study of general shells.

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